A Deeper Look at a Calculus I Activity

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June 27, 2016

"It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the ensemble and the details."

Figure: ?

"Thought is only a flash between two long nights, but this flash is everything."

The Necessity Principle

"For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to **intellectual need**, not social or economic need."

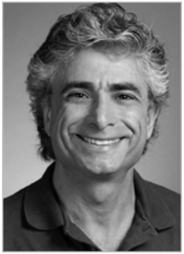
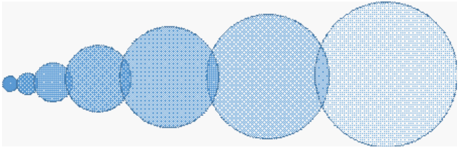


Figure: Guershon Harel

FLOCK-inspired Math 75 Redesign Features

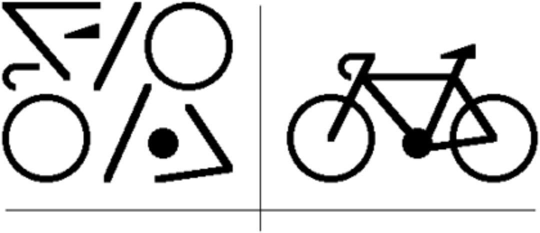
- Three lecture/problem solving days (MTW) and 1 active learning day (TH).
- Sequencing of topics based upon the 'Wholecept Resolution' perspective as much as possible.
- Active Learning 'Tactivities' mostly from http://math.colorado.edu/activecalc/
- Designated class periods having students working at boards together on 12 group quizzes (gallery format).
- Students encouraged to attend Calculus Success weekly $1\frac{1}{2}$ hour sessions.



Unilinear Concept Formation

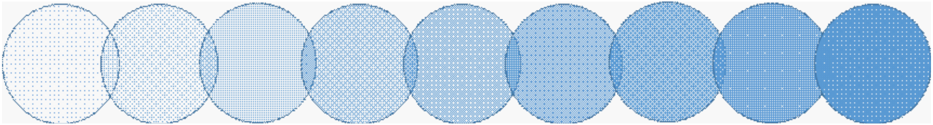
'Unilinear - developing or arranged serially and predictably, without deviation'

The unified whole is different from the sum of the parts.



Definition

A Wholecept *is a cognitive* structure, arrangement, or pattern of mathematical phenomena so integrated as to constitute a functional unit with properties not derivable by summation of its parts.



'Wholecept' Resolution

Difference Quotient Wholecept introduction on day 1

... review for the sake of review is discouraged!

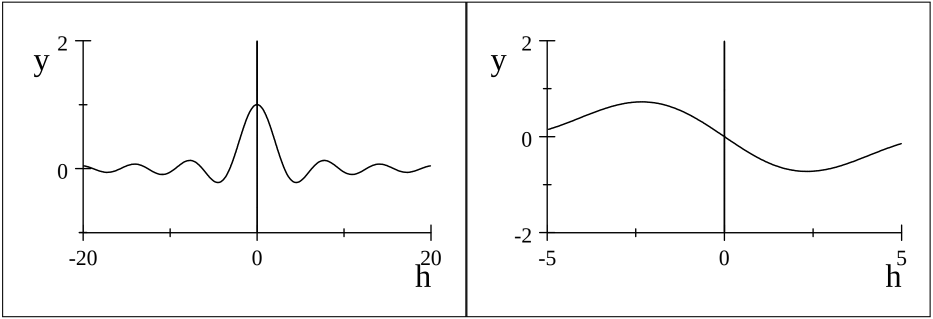
Students need to know 'why' they are presented with this knowledge, so it's best to tell them asap. [The Necessity Principle]

Week	*M*	*T*	*W*	*TH-Tactivities*
1	1.1	1.2	1.3	Function Placemats
2	1.4	2.2	*gps-1*	Transformation Matching
3	2.3	2.4	*gps-2*	Limit Sentences
4	2.5	2.6	*gps-3*	**Exam 1**
5	3.1-3.2	3.3-3.4	**3.7-3.8**	Graphical Limit Laws
6	3.5/3.7	3.9/3.7	3.10/3.7	Definition of Derivative
7	*3.11*	*3.11*	*3.11*	Related Rates Solitaire
8	3.6	*gps-4*	*gps-5*	**Exam 2**
9	4.1	4.2	4.3	Derivative Matching Cards
10	4.4	4.5	*gps-6*	Grade this Quiz
11	4.6	4.7	*gps-7*	Sketching Snippets
12	4.8	4.9	*gps-8*	**Exam 3**
13	5.1	5.2	*gps-9*	Wacky Limits
14	5.3-5.4	5.5	*gps-10*	Definite Integral Dominoes
15	*5.5*	*5.5*	*gps-11*	Cups
16	*5.5*	*5.5*	*gps-12*	**Exam 4**

Text: Calculus Early Transcendentals, 2nd ed., Briggs, Cochran & Gillett, Pearson.

Calculus I Wholecepts to Teach Early with Repetition

- Difference Quotient & Secant Slope as Average
- Parent Graphs & Function Transformations
- Limits & Continuity
- The Derivative as a two-sided limit
- Function Composition & the Chain Rule
- Implicit Differentiation
- General Problem Solving Strategies



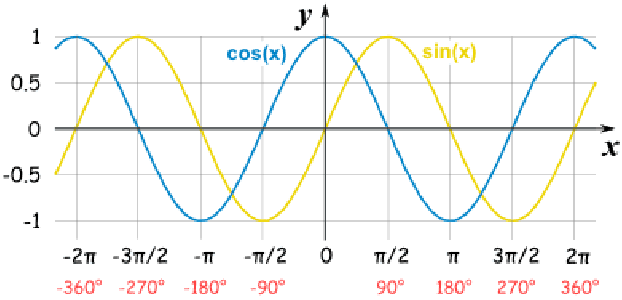
$$\lim_{h\to 0}\frac{\sin(h)}{h}=1 \qquad \lim_{h\to 0}\frac{\cos(h)-1}{h}=0$$

$$(\sin x)'=\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}=\lim_{h\to 0}\frac{\sin x\cos(h)+\sin(h)\cos x-\sin x}{h}=$$

$$\lim_{h\to 0}\frac{\sin x(\cos(h)-1)+\sin(h)\cos x}{h}$$

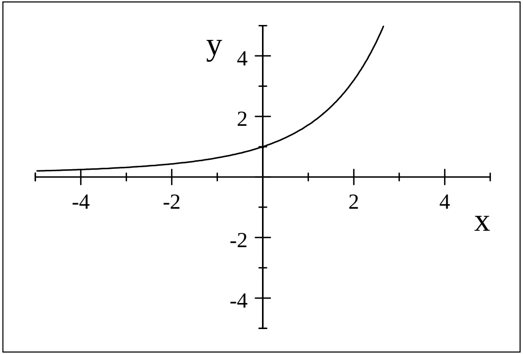
$$\lim_{h\to 0}\sin x\cdot\frac{\cos(h)-1}{h}+\cos x\cdot\frac{\sin(h)}{h}=\sin x\cdot 0+\cos x\cdot 1=\cos x.$$

- The Chain Rule rules!
$(\cos x)' = (\sin(x + \frac{\pi}{2}))' = \cos(x + \frac{\pi}{2}) \cdot \frac{d(x+\frac{\pi}{2})}{dx} = -\sin x.$



$$(\sin x)' = (\cos(x - \frac{\pi}{2}))' = -\sin(x - \frac{\pi}{2}) = \cos x$$

$(e^x)'$



$$\lim_{h\to 0}\frac{e^h-1}{h}=1$$

$$(e^x)'=\lim_{h\to 0}\frac{e^{(x+h)}-e^x}{h}=\lim_{h\to 0}\frac{e^x(e^h-1)}{h}=e^x$$

$(lnx)\prime$ and $(x^n)'$

- Chain Rule $\longrightarrow$Implicit Differentiation

$y = \ln x \longrightarrow e^y = x \longrightarrow$

$$(e^y)' = e^y \cdot y^{'} = 1 \longrightarrow y^{'} = \frac{1}{e^y} = \frac{1}{x}$$

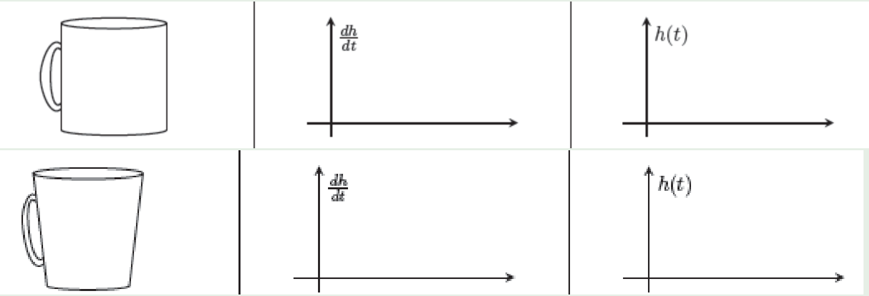
$y = x^n \longrightarrow \ln y = \ln(x^n) \longrightarrow$
$\ln y = n \ln x \longrightarrow \frac{1}{y} \cdot y' = \frac{n}{x} \longrightarrow$
$y' = \frac{n}{x} \cdot y = n\frac{x^n}{x} = nx^{n-1}.$

Activity Description

Problem

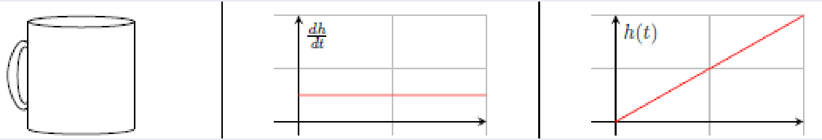
Coffee is being poured at a constant rate v into coffee cups of various shapes. Sketch rough graphs of the rate of change of the depth $h'(t)$ and of the depth $h(t)$ as a functions of time t.

Example



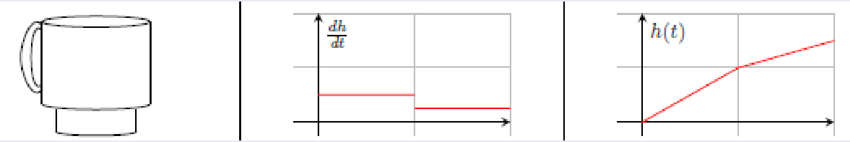
Most students produce graphs like this:

Solution



Students also tend to negotiate this type of cup:

Solution (two stacked-cylinders)

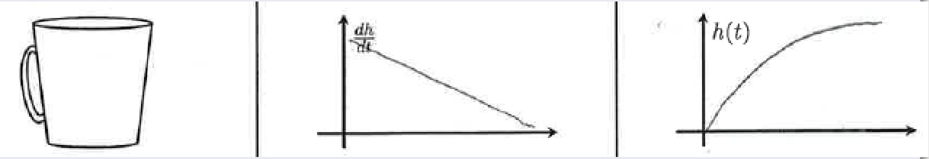


But for this cup ...

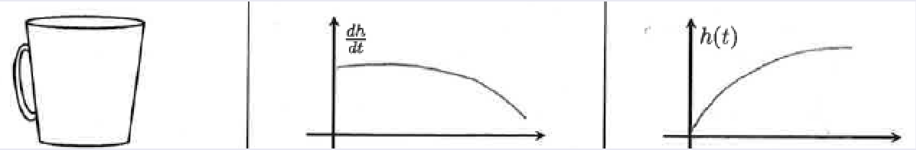
Example (inverted frustrum)

Two types of student-solutions occurred about 75% of the time last semester:

Solution (linear rate decrease)



Solution (concave down rate decrease)



Now for a deeper look ...

Example

Back to the cylindrical cup with base radius r_0, we can safely conclude that since the *volume* $V(t)$ of coffee in the cup increases at a constant rate, then so does its *depth*.

Hence, $h'(t) \equiv h$ and $h(t) = ht$ (the cup being empty initially, i.e., $h(0) = 0$).

$$V(t) = \pi r_0^2 h(t).$$

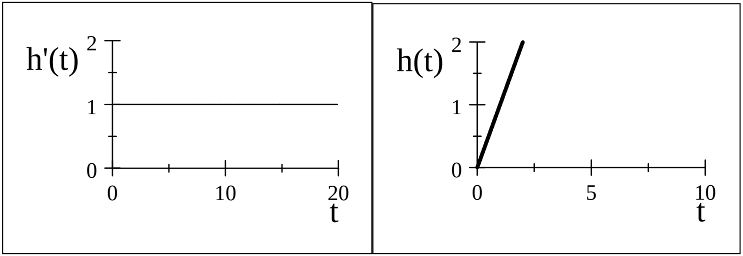
Differentiating both sides relative to t

$$V'(t) = \pi r_0^2 h'(t)$$

and considering that $V'(t) = v$, we have: $h'(t) = \dfrac{v}{\pi r_0^2} \longmapsto h(t) = \dfrac{v}{\pi r_0^2} t$ (given $h(0) = 0$).

- Observe that $h'(t)$ is not the same as $V'(t)$.

Letting $v = \pi r_0^2$ satisfies the initial conditions and produces the following graphs for $h(t)$ and $h'(t)$:



And looking more closely at the inverted frustrum cup ...

Example

Let $r(t)$ be the radius of the surface of coffee. Then

$$r(t) = r_0 + mh(t)$$

with some $m > 0$.

In this case, it appears "natural" to think of $h'(t)$ as a linear function based on the linear dependence of the radius $r(t)$ on the depth $h(t)$ which leads to the conclusion that $h'(t)$ is a *linear function* and $h(t)$ is *quadratic.* But as we shall see, this error in qualitative reasoning fails the test by mathematics ...

Frustrum cup analysis

By the conical frustum volume formula, the volume of coffee in the cup at time t is given by:

$$V(t) = \frac{1}{3}\pi[r_0^2 + r_0 r(t) + r^2(t)]h(t)$$

Instead of differentiating both sides of the above equation relative to t, which would make things more convoluted, we consider that $V'(t) \equiv v$ immediately implies $V(t) = vt$ (with $V(0) = 0$); hence, $h(t)$ is to be found from the *cubic equation:*

$$m^2 h^3(t) + 3mr_0 h^2(t) + 3r_0^2 h(t) - 3vt/\pi = 0.$$

The *general formula for the roots* of such an equation in this case yields $h(t)$ explicitly as:

$$h(t) = -\frac{1}{3m^2}\left[3mr_0 + \sqrt[3]{-27m^3 r_0^3 - 81m^4 vt/\pi}\right].$$

Frustrum cup analysis contd.

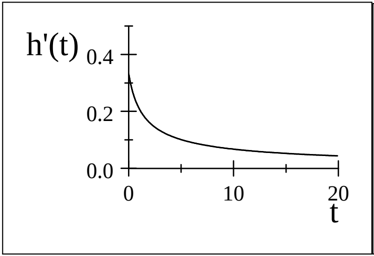
Hence,

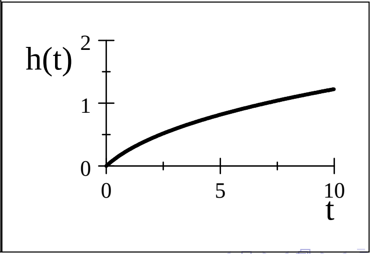
$$h(t) = a(t+b)^{1/3} + c$$

with some $a, b > 0$ and $c < 0$ such that $h(0) = ab^{1/3} + c = 0$ and

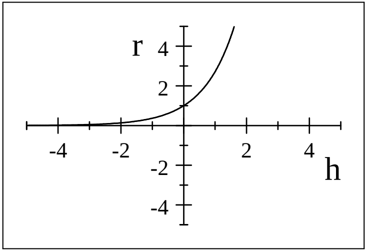
$$h^{'}(t) = \frac{a}{3}(t+b)^{-2/3}.$$

Letting $a = b = 1$ and $c = -1$ satisfies the initial conditions and produces the following graphs for $h(t)$ and $h'(t)$:





Now for a cup with Exponential shaped - sides ...



Using the disk-method from 0 to h

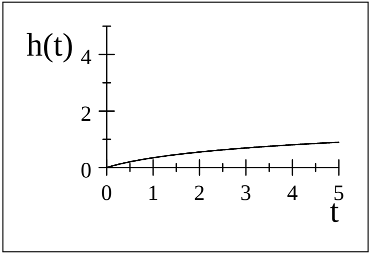
$$V(t) = \pi \int_0^h (e^x)^2 \, dh = \frac{\pi e^{2h}}{2} - \frac{\pi}{2} = vt.$$

Solving for h

$$h(t) = \frac{1}{2} \cdot \ln\left(\frac{2vt}{\pi} + 1\right)$$

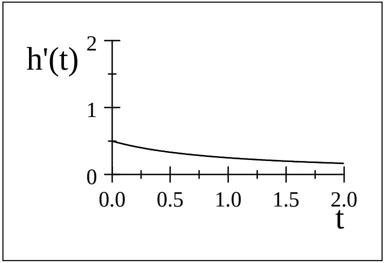
WLOG, let $v = \frac{\pi}{2} ml/\sec$

$$h(t) = \frac{1}{2} \cdot \ln(t+1)$$



Differentiating both sides relative to t

$$h'(t) = \frac{1}{2(t+1)}$$



Reflection

Why must it be the case that the $h'(t)$ graph MUST be concave up?

- Any thoughts?

Anticipated Responses

- In graphing the derivative based on the $h(t)$ graph, the slopes are positive and become less positive tending to 0.
- If $h'(t)$ had a concave down graph, the height would stop at some point, and go backwards.
- Also if concave down, this would produce non-sensical anti-derivative graphs (more on this later).

All of this aside, how do we get students in a position to make qualitatively correct $h'(t)$ graphs?

- This semester Cups will be done in Week 15 instead of Week 9
- Changes being made as to how *Tactivities* are assessed
$\longrightarrow$ *photos* – *reflections* – *prompts*–e-portfolios!
- Required for this assignment now are three screen shots added to the portfolio from a Geogebra derivative sketching activity, shared with FLOCK by Dr. Agnes Tuska.

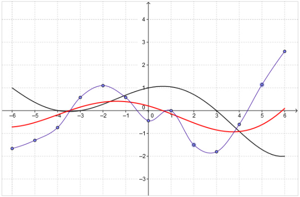
Try to Graph the Derivative...

webspace.ship.edu/msrenault/GeoGebraCalculus/derivative_try_to_graph.html

geogebra derivative sketching activity

Try to Graph the Derivative Function

HELP



Reset the graph

Check accuracy 46

Show results! Final accuracy: 43%

You are given the graph of $f(x)$, and your task is to show what $f'(x)$ looks like.

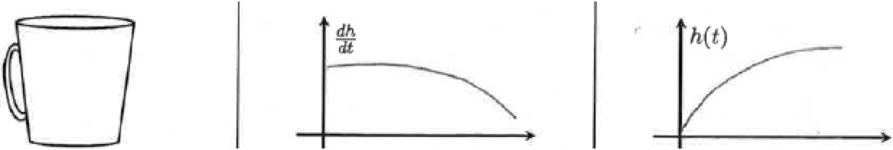
Explore

1. The graph of $f(x)$ is shown in black. Drag the blue points up and down so that together they follow the shape of the graph of $f'(x)$.
2. When you think you have a good representation of $f'(x)$, click the "Show results!" button below the applet. This reveals the true graph of $f'(x)$, drawn in red.
3. You can continue to move points and see how the accuracy changes.
4. Click "Reset the graph" to get a new problem!

This page is part of the GeoGebra Calculus Applets project.

Example

What are some familiar functions you know that look like your $h(t)$ graphs? After discussion with your table, write a reflection paragraph on their derivative graphs in the context of the problem.



- $y = \sqrt{x}$

Example

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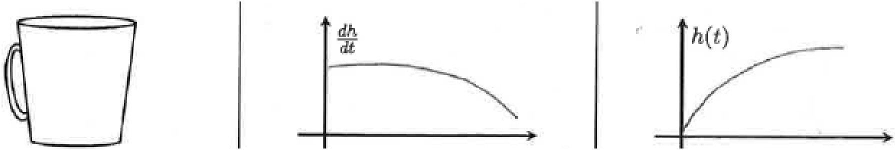
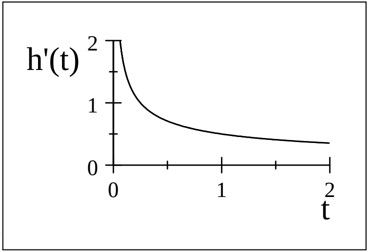
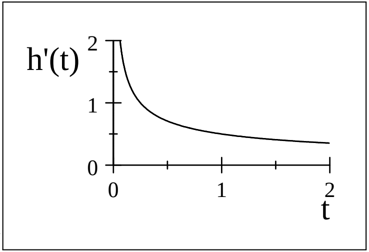


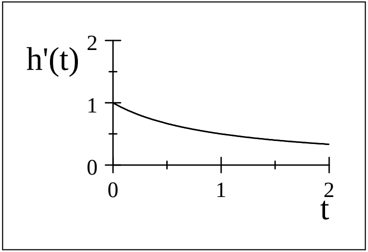
- $y = \sqrt{x}$
- $y = \ln(x+1)$

- $h'(t) = \frac{1}{2\sqrt{t}} \rightarrow$ 

... what about this graph?

- $h'(t) = \frac{1}{2\sqrt{t}} \rightarrow$ 

... what about this graph?

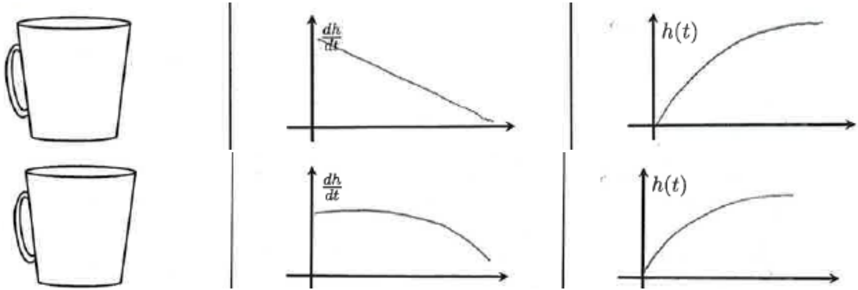
- $h'(t) = \frac{1}{t+1} \rightarrow$ 

... this is better. Why?

Example

What are some familiar functions you know that look like your $h'(t)$ graphs? After discussion with your table, write a reflection paragraph on their anti-derivative graphs in the context of the problem.

(be sure your anti-derivative satisfies $h(0) = 0$)!

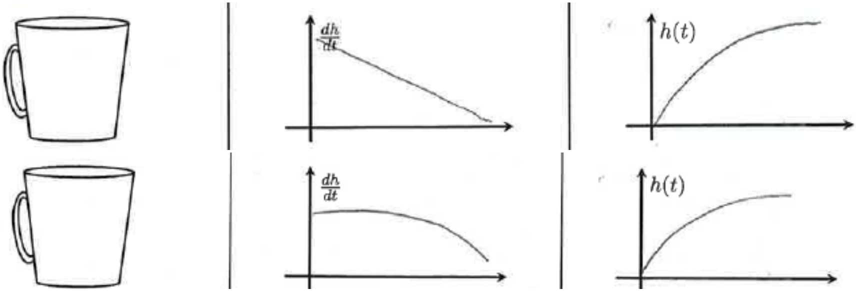


- $h'(t) = -2t + 3$

Example

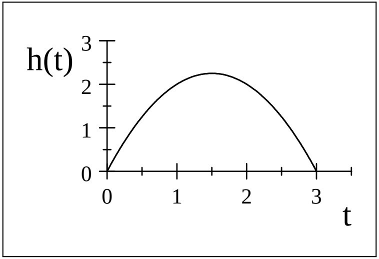
What are some familiar functions you know that look like your $h'(t)$ graphs? After discussion with your table, write a reflection paragraph on their anti-derivative graphs in the context of the problem.

(be sure your anti-derivative satisfies $h(0) = 0$)!

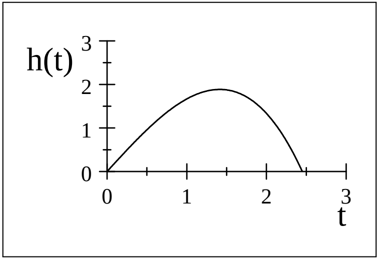


- $h'(t) = -2t + 3$
- $h'(t) = -t^2 + 2$

$\int -2t + 3 \; dt \rightarrow -t^2 + 3t + C \rightarrow C = 0$
$\rightarrow h(t) = -t^2 + 3t$



$$\int -t^2 + 2\ dt \rightarrow -\frac{t^3}{3} + 2t + C \rightarrow C = 0$$
$$\rightarrow h(t) = -\frac{t^3}{3} + 2t$$



Thank you for coming to my talk and in particular I would like to thank ...

- My FLOCK colleagues Marat Markin (co-author), Agnes Tuska, Kay Kelm, Travis Kelm, and Comlan de Souza for the many valuable interactions and collaborations.
- Dr. David Webb and the BOALA group for sharing activities fundamental to our reform efforts at Fresno State.
- And finally to the organizers of MTEP for making all of this possible!